JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **25**, No. 4, November 2012

FINITELY GENERATED gr-MULTIPLICATION MODULES

Seungkook Park*

Abstract. In this paper, we investigate when gr-multiplication modules are finitely generated and show that if M is a finitely generated gr-multiplication R-module then there is a lattice isomorphism between the lattice of all graded ideals I of R containing $\text{ann}(M)$ and the lattice of all graded submodules of M.

1. Introduction

Let R be a commutative ring with identity $1 \neq 0$ and M a unital R-module. M is called a *multiplication module* module provided for each submodule N of M, there exists an ideal I of R such that $N = IM$ [2]. Let G be a multiplicative group with identity e. A ring R is said to be a *graded ring of type* G if there is a family of additive subgroups of R, say $\{R_i \mid i \in G\}$, such that $R = \bigoplus_{i\in G} R_i$ and $R_iR_j \subseteq R_{ij}$ for all $i,\ j \in G$, where R_iR_j is the set of all finite sums of products r_ir_j with $r_i \in R_i$ and $r_j \in R_j$. The elements of $h(R) = \bigcup_{i\in G} R_i$ are called the homogeneous elements of R. Any nonzero $r \in R$ has a unique expression as a sum of homogeneous elements, that is, $r = \sum_{i\in G} r_i$ where r_i is nonzero for a finite number of i in G. The nonzero elements r_i in the decomposition of r are called the homogeneous components of r. Let R be a graded ring of type G. An R-module M is said to be a *graded R-module* if there is a family $\{M_i \mid i \in G\}$ of additive subgroups of M such that $M = \bigoplus_{i\in G} M_i$ and $R_iM_j \subseteq M_{ij}$ for all $i,\ j \in G$. Elements of $h(M) = \bigcup_{i\in G} M_i$ are called the homogeneous elements of M. A submodule N of M is a graded submodule if $N = \bigoplus_{i\in G}(N \cap M_i)$, or equivalently, if for any

Received August 03, 2012; Accepted October 10, 2012.
2010 Mathematics Subject Classification: Primary 16W50; Secondary 13A02.
Key words and phrases: finitely generated gr-multiplication module, multiplication module.
*This research was supported by the Sookmyung Women's University Research Grants 2011.

$x \in N$, the homogeneous components of x are again in N. Properties of finitely generated multiplication module have been studied in [2], [3], [6], and [7]. In this paper, we study the properties of finitely generated gr-multiplication modules and investigate when gr-multiplication modules are finitely generated.

2. Main results

DEFINITION 2.1. Let R be a graded ring and let M be a graded R-module. Then M is called a gr-multiplication module if for any graded submodule N of M, there exists a graded ideal I of R such that $N = IM$.

REMARK 2.2. [4] Let R be a graded ring. If M is a graded R-module and N is a submodule of M, then $(N : M)$ is a graded ideal of R, where $(N : M) = \{r \in R \mid rM \subseteq N\}$.

For any graded submodule N of M, we denote $(N : M)_g$ the graded ideal of R generated by $(h(N) : h(M)) = \{r \in h(R) \mid rh(M) \subseteq h(N)\}$. Note that $(N : M)_g$ is the graded ideal of R generated by $(N : M) \cap h(R)$ and that $(N : M)_g = (N : M)$.

The following lemma can be found in [1] and also in [8].

LEMMA 2.3. *Let R be a graded ring. Let M be a finitely generated graded R-module and let I be a graded ideal of R such that $M = IM$. Then there exists $q \in I$ such that $(1 - q)M = 0$.*

Proof. See Lemma 2.1 in [8] □

DEFINITION 2.4. An R-module M is faithful if, whenever $r \in R$ is such that $rM = 0$, then $r = 0$.

THEOREM 2.5. *Let R be a graded ring and M a faithful gr-multiplication R-module. Then the following statements are equivalent.*

(i) *M is finitely generated.*
(ii) *If A and B are graded ideals of R such that $AM \subseteq BM$ then $A \subseteq B$.*
(iii) *For each graded submodule N of M, there exits a unique graded ideal I of R such that $N = IM$.*
(iv) *$M \neq AM$ for any proper graded ideal A of R.*
(v) *$M \neq PM$ for any gr-maximal ideal P of R.*

Proof. (i) $\Longrightarrow$ (ii).
Suppose that M is finitely generated. Let A and B be graded ideals of

R such that $AM \subseteq BM$. Let $a \in h(A)$. Let $K = \{r \in R \mid ra \in B\}$. Then K is a graded ideal of R. Suppose $K \neq R$. Then there exists a gr-maximal ideal P of R such that $K \subseteq P$. Suppose $M = PM$. By Lemma 2.3, $(1-p)M = 0$ for some $p \in P$. Since M is faithful, $p = 1$, which is a contradiction. Thus $M \neq PM$. Let $m \in h(M)$ with $m \notin PM$. Then there exists a graded ideal I of R such that $Rm = IM$. If $I \subseteq P$ then $Rm = IM \subseteq PM$ and hence $m \in PM$, which is a contradiction. Therefore $I \nsubseteq P$. Since $R = P + I$, $1 = q + i$ for some $q \in P$ and $i \in I$. Hence $1 - q \in I$. Thus $(1-q)M \subseteq IM = Rm$. In particular, $(1-q)am \in (1-q)BM = B(1-q)M \subseteq Bm$. Thus there exists $b \in B$ such that $[(1-q)a - b]m = 0$. Since $(1-q)\operatorname{ann}(m)M = \operatorname{ann}(m)(1-q)M \subseteq \operatorname{ann}(m)Rm = 0$, $(1-q)\operatorname{ann}(m) \subseteq \operatorname{ann}(M) = 0$. Thus $(1-q)[(1-q)a - b] = 0$ and this implies that $(1-q)^2 a = (1-q)b \in B$ so that $(1-q)^2 \in K \subseteq P$, which is a contradiction. This contradiction leads us to the conclusion that $K = R$ and hence $a \in B$. It follows that $A \subseteq B$.

(ii) $\Longrightarrow$ (iii).

Let N be a graded submodule of M. Suppose that I is a graded ideal of R such that $N = IM$. Since $I \subseteq (N : M) = (N : M)_g$, $N = IM \subseteq (N : M)_g M \subseteq N$. Thus $N = (N : M)_g M$. Then $(N : M)_g M = IM$ and hence $(N : M)_g = I$ by (ii).

(iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v).

These are trivial.

(v) $\Longrightarrow$ (i).

Let P be a gr-maximal ideal of R. Then $M \neq PM$. So there exists $m \in h(M)$ with $m \notin PM$. Then $Rm = BM$ for some graded ideal B of R. Clearly $B \nsubseteq P$. Thus $(Rm : M) \nsubseteq P$ and $(Rm : M)$ is a graded ideal of R. It follows that $R = \sum_{m \in h(M)} (Rm : M)$. There exists a positive integer n and elements $m_i \in h(M)$, $r_i \in (Rm_i : M)$ such that $1 = r_1 + \cdots + r_n$. If $x \in M$, then $x = r_1 x + \cdots + r_n x \in Rm_1 + \cdots + Rm_n$. If follows that $M = Rm_1 + \cdots + Rm_n$. $\square$

Corollary 2.6. *Let R be a graded ring. If M is a finitely generated gr-multiplication R-module then there is a lattice isomorphism ϕ from the lattice of all graded ideals I of R containing* $\operatorname{ann}(M)$ *and the lattice of all graded submodules of M given by $\phi(I) = IM$.*

Let R be a graded ring and let M be a graded R-module. Let $\mathcal{T}$ denote the collection of graded ideals I of R such that $M = IM$, and $\tau(M)$ the intersection of all graded ideals in $\mathcal{T}$. Then $\tau(M)$ is a graded ideal of R. The following corollary is a restatement of Theorem 2.5

Corollary 2.7. *Let R be a graded ring. A faithful gr-multiplication R-module M is finitely generated if and only if $\tau(M) = R$.*

Lemma 2.8. *Let R be a graded ring and let M be a faithful gr-multiplication R-module. Then $M = \tau(M)M$.*

Proof. Let $\mathcal{T}$ be the collection of graded ideals I_λ $(\lambda \in \Lambda)$ of R such that $M = I_\lambda M$. Then $\tau(M) = \cap_{\lambda\in\Lambda} I_\lambda$. Let $x \in h(M) = h\left(\cap_{\lambda\in\Lambda}(I_\lambda M)\right)$ and let $K = \{r \in R \mid rx \in \tau(M)M\}$ be a graded ideal of R. Suppose $K \neq R$. Then there exists a gr-maximal ideal P of R such that $K \subseteq P$. Suppose that $M = PM$. Since $Rx = AM$ for some graded ideal A of R, we have $Rx = AM = APM = PAM = Px$ and $x = px$ for some $p \in P$. Thus $(1-p)x = 0$. Hence $1 - p \in K \subseteq P$, which is a contradiction. Thus $M \neq PM$. Let $m \in h(M)$ with $m \notin PM$. Then there exists a graded ideal B of R such that $Rm = BM$. If $B \subseteq P$ then $Rm = BM \subseteq PM$ and hence $m \in PM$, which is a contradiction. Therefore $B \nsubseteq P$. Since $R = P+B$, $1 = q+b$ for some $q \in P$ and $b \in B$. Hence $1 - q \in B$. Thus $(1-q)M \subseteq BM = Rm$. Then $(1-q)x \in (1-q)I_\lambda M = I_\lambda(1-q)M \subseteq I_\lambda m$ for all $\lambda \in \Lambda$. Thus $(1-q)x \in \cap_{\lambda\in\Lambda}(I_\lambda m)$. For each $\lambda \in \Lambda$, there exists $a_\lambda \in I_\lambda$ such that $(1-q)x = a_\lambda m$. Choose $\alpha \in \Lambda$. For each $\lambda \in \Lambda$, $a_\alpha m = a_\lambda m$ so that $(a_\alpha - a_\lambda)m = 0$. Now $(1-q)(a_\alpha - a_\lambda)M = (a_\alpha - a_\lambda)(1-q)M \subseteq (a_\alpha - a_\lambda)R_m = 0$ implies $(1-q)(a_\alpha - a_\lambda) = 0$. Therefore $(1-q)a_\alpha = (1-q)a_\lambda \in I_\lambda$ $(\lambda \in \Lambda)$ and hence $(1-q)a_\alpha \in \tau(M)$. Thus $(1-q)^2 x = (1-q)a_\alpha m \in \tau(M)M$. It follows that $(1-q)^2 \in K \subseteq P$, which is a contradiction. Thus $K = R$ and $x \in \tau(M)M$. Hence $h(M) = h\left(\cap_{\lambda\in\Lambda}(I_\lambda M)\right) \subseteq \tau(M)M$. This shows that $M \subseteq \tau(M)M$ and hence $M = \tau(M)M$. □

A graded ideal P of R (i.e., a graded R-submodule of R) is called *gr-prime* if $P \neq R$ and whenever $rs \in P$ $(r, s \in h(R))$ then $r \in P$ or $s \in P$.

Lemma 2.9. *Let R be a graded ring. Let M be a faithful gr-multiplication R-module and $T = \tau(M)$. Then we have*

(i) *$m \in Tm$ for each $m \in h(M)$,*
(ii) *$T = T^2$,*
(iii) *$T \subseteq P$ or $R = T + P$ for every gr-prime ideal P of R.*

Proof. (i), (ii) See Lemma 2.6 in [1]
(iii) Let P be a gr-prime ideal of R. If $M = PM$ then $T \subseteq P$. Suppose $M \neq PM$, then there exists $m \in h(M)$ such that $m \notin PM$. So $Rm = IM$ for some graded ideal I of R. Clearly $I \nsubseteq P$. By (i), there exists $t \in T$ such that $(1-t)m = 0$ and hence $(1-t)IM = (1-t)Rm = 0$.

Since M is faithful, $(1-t)I = 0 \in P$ and hence $1-t \in P$, that is, $R = T + P$. $\square$

The following lemma can be found in [5].

Lemma 2.10. *If I is a finitely generated idempotent ideal of a commutative ring R, then I is principal and is generated by an idempotent element*

Theorem 2.11. *Let R be a graded ring. A faithful gr-multiplication R-module M is finitely generated if and only if $\tau(M)$ is finitely generated.*

Proof. Let $T = \tau(M)$. If M is finitely generated, then $T = R$ by Corollary 2.7. Thus T is finitely generated.

Conversely, suppose that T is finitely generated. Since M is a faithful gr-multiplication R-module, $T = T^2$. By Lemma 2.10, $T = Re$ for some idempotent element e of R. By Lemma 2.8, $M = TM$. Then $(1-e)M = (1-e)TM = TM - eTM = ReM - Re^2M = 0$ and hence $1-e = 0$. Therefore $T = R$. By Corollary 2.7, M is a finitely generated R-module. $\square$

From Corollary 2.7 and Theorem 2.11 we have that for a graded ring R and a faithful gr-multiplication R-module M, the following are equivalent :

(i) M is finitely generated.
(ii) $\tau(M) = R$.
(iii) $\tau(M)$ is finitely generated.

Theorem 2.12. *Let M be a faithful gr-multiplication R-module. Then M is finitely generated if and only if $M \neq PM$ for all minimal gr-prime ideals P of R.*

Proof. The necessity is an immediate consequence of Theorem 2.5.

Conversely, suppose that M is not finitely generated. By Corollary 2.7, $T = \tau(M) \neq R$. Let Q be a gr-maximal ideal of R such that $T \subseteq Q$ and let P be a minimal gr-prime of R such that $P \subseteq Q$. By Lemma 2.9 (iii), $T + P \subseteq Q$ implies that $T \subseteq P$ and hence $M = PM$. $\square$

Theorem 2.13. *Let R be a graded ring and let M be a gr-multiplication R-module. If $R = I + \mathrm{ann}(M)$ for every graded ideal I of R with $M = IM$, then M is a finitely generated.*

Proof. Let $M = \bigoplus_{i \in G} M_i$ and let $h(M) = \bigcup_{i \in G} M_i = \{m_\lambda \mid \lambda \in \Lambda\}$. Then $M = \sum_{\lambda \in \Lambda} Rm_\lambda$. Since M is a gr-multiplication module, for each Rm_λ, there exists a graded ideal I_λ of R such that $Rm_\lambda = I_\lambda M$. Let $I = \sum_{\lambda \in \Lambda} I_\lambda$. Then

$$M = \sum_{\lambda \in \Lambda} Rm_\lambda = \sum_{\lambda \in \Lambda} I_\lambda M = IM.$$

By assumption, we have $R = I + \mathrm{ann}(M)$ and hence $1 \in I + \mathrm{ann}(M)$. Then there exists a finite subset $\{I_{\lambda_1}, \ldots, I_{\lambda_n}\}$ of the set $\{I_\lambda\}_{\lambda \in \Lambda}$ such that $1 \in \sum_{j=1}^n I_{\lambda_j} + \mathrm{ann}(M)$. Then

$$M = 1 \cdot M \subseteq (\sum_{j=1}^n I_{\lambda_j} + \mathrm{ann}(M))M = \sum_{j=1}^n I_{\lambda_j} M = \sum_{j=1}^n Rm_{\lambda_j}.$$

Therefore M is finitely generated. □

Lemma 2.14. *Let R be a graded ring and let M be a graded R-module. If X and Y are two graded submodules of M such that $X + Y$ is a gr-multiplication R-module, then*

$$(X + Y) \cap Z = (X \cap Z) + (Y \cap Z)$$

for any graded submodule Z of M.

Proof. Since $(X+Y) \cap Z$ is a graded submodule of a gr-multiplication module $X + Y$, there exists a graded ideal I such that $(X + Y) \cap Z = I(X + Y)$. Since $IX \subseteq X \cap Z$ and $IY \subseteq Y \cap Z$, we have

$$\begin{aligned}(X + Y) \cap Z &= I(X + Y) = IX + IY \\ &\subseteq (X \cap Z) + (Y \cap Z) \subseteq (X + Y) \cap Z.\end{aligned}$$

□

Theorem 2.15. *Let R be a graded ring and let M be a graded R-module. If X and Y are two graded submodules of M such that $X + Y$ is a finitely generated gr-multiplication R-module, then*

$$R = (X : Y) + (Y : X).$$

Proof. Since $X + Y$ is finitely generated, there exist a positive integer n and elements $x_i \in h(X)$, $y_i \in h(Y)$ $(1 \le i \le n)$ such that $X + Y = \sum_{i=1}^n R(x_i + y_i)$. Let $y \in Y$. By Lemma 2.14, for any $1 \le i \le n$, we have

$$R(x_i + y) = R(x_i + y) \cap (X + Y) = (R(x_i + y) \cap X) + (R(x_i + y) \cap Y).$$

Thus, there exist elements $r \in R$ and $z \in Y$ such that

$$(x_i + y) = r(x_i + y) + z \text{ and } r(x_i + y) \in X.$$

Thus, $(1-r)x_i = (r-1)y + z \in Y$ and $ry \in X$. Then $(1-r) \in (Y : Rx_i)$ and $r \in (X : Ry)$. It follows that

$$1 = r + (1-r) \in (X : Ry) + (Y : Rx_i) \ (1 \leq i \leq n)$$

and hence

$$R = (X : Ry) + (Y : Rx_i) \ (1 \leq i \leq n).$$

Thus

$$R = (X : Ry) + ((Y : Rx_1) \cap \cdots \cap (Y : Rx_n)) = (X : Ry) + (Y : X).$$

In particular,

$$R = (X : Ry_i) + (Y : X) \ (1 \leq i \leq n).$$

Therefore,

$$R = ((X : Ry_1) \cap \cdots \cap (X : Ry_n)) = (X : Y) + (Y : X).$$

□

References

[1] S. E. Atani and R. E. Atani, *Graded multiplication modules and the graded ideal $\theta_g(M)$*, Turkish J. Math. **35** (2011), no. 1, 1–9.

[2] A. Barnard, *Multiplication modules*, J. Algebra. **71** (1981), no. 1, 174–178.

[3] Z. A. El-Bast and P.F. Smith, *Multiplication modules*, Comm. Algebra. **16** (1988), no. 4, 755–779.

[4] J. Escoriza and B. Torrecillas, *Multiplication objects in commutative Grothendieck categories*, Comm. Algebra. **26** (1998), no. 6, 1867–1883.

[5] R. Gilmer, *An existence theorem for non-Noetherian rings*, Amer. Math. Monthly. **77** (1970), 621–623.

[6] A. G. Naoum, B. A. Al-Hashimi, and H. R. Hassan, *Maximal, prime and primary submodules of finitely generated multiplication modules*, Dirāsāt Ser. B Pure Appl. Sci. **18** (1991), no. 1, 7–17.

[7] K. Nekooei, *On finitely generated multiplication modules*, Czechoslovak Math. J. **55(130)** (2005), no. 2, 503–510.

[8] N. Zamani, *Finitely generated graded multiplication modules*, Glasg. Math. J. **53** (2011), no. 3, 693–705.

*

Department of Mathematics
Sookmyung Women's University
Seoul 140-742, Republic of Korea
E-mail: skpark@sookmyung.ac.kr